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Derks, J.; Tijs, S.H.

Published in:
International Game Theory Review

Publication date:
2000

Citation for published version (APA):
Derks, J., & Tijs, S. H. (2000). On merge properties of the Shapley value. *International Game Theory Review*, *2*(4), 249-257.

International Game Theory Review, Vol. 2, No. 4 (2000) 249–257

ON MERGE PROPERTIES OF THE SHAPLEY VALUE

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Given a transferable utility game, where the players merge into subgroups described by a partition, we address the following question: under which conditions on the characteristic function and partition, merging is beneficial if the Shapley value is applied. Our results can be positioned among the search for well-defined classes of games where merging of players is possible without utility loss in case the Shapley value is chosen as the outcome of the game, and we will report on two of these classes of games arising from telecommunication problems.

1. Introduction

Recently, some attention is paid to merging, or amalgamation, of players in the context of transferable utility games. Two types of fusion evolve depending on how the merging players are treated in the resulting game. Haller (1994) discusses the situation where any two players are allowed to make binding agreements, and depending on these agreements, a new game evolves from the original game, with the same player set. In Lehrer (1988), two players are allowed to merge into one player so that the player set of the resulting game is reduced. In both cases the players are fully aware of how the actual game is evaluated, and the players are, therefore, able to decide whether or not merging is profitable.

Haller considers the Shapley value as the evaluation method, and he concludes that merging is not profitable in general, but interesting subclasses of games exist where merging is profitable. Lehrer applies the Banzhaf value. He shows that it is always profitable to merge, and he uses this property in an axiomatic framework for the Banzhaf value. See Nowak (1997) for a related characterization of the Banzhaf value.

This note considers the Lehrer type of merging. We, however, assume that the Shapley value is used for evaluation of the games. So, before the game is actually evaluated, the players are allowed to form disjoint groups, the merging coalitions, which are assumed to act as one player. The resulting game is the so-called quotient game with respect to the partition of the player set into the merging coalitions and the non-merging players. If the Shapley value of a merging coalition is at least the sum of the Shapley values of its players in the original game, then the merging of these players is profitable. We address the question under which conditions, both on the characteristic function of the game and the partition of the player set into merging coalitions, some or all merging coalitions profit from merging.

Our main result roughly states that a coalition of players profits from merging if for each coalition with a positive (negative) dividend the number of merging players in that coalition is smaller (larger) than "average". The intuition behind this result is the fact that the Shapley value distributes the dividends of the coalitions equally among its members. Merging of players belonging to a coalition with negative dividend may therefore be profitable, whereas players belonging to a coalition with positive dividend would like to stay non-merged in order to receive their positive portion of the dividend, without sharing this with others.

Normally, merging players are unaware of the merging behaviour of the other players, so that there is uncertainty of which quotient game is actually played. One way to deal with this problem is to assume that the remaining players do not merge. A coalition is called mergeable if it is profitable for its players to merge, given that they are the only players that merge. We will derive some results on mergeable coalitions, and we will discuss the existence of mergeable coalitions in several classes of games. Two of these classes arise from telecommunication problems [Gibbens *et al.* (1991) and Van den Nouweland *et al.* (1996)].

The notion of mergeable coalitions is also addressed, although implicitly, by Vázquez-Brage *et al.* (1997) in the context of airport cost games, with the Owen value as evaluation criterion.

The set-up of the paper is as follows. Section 2 introduces the elementary definitions and notations. In Sec. 3, we introduce the notions with respect to merging, and the main results are stated and proven. In Sec. 4, we discuss several practical instances and applications in which mergeable coalitions are observed.

2. Preliminaries

As stated previously, we will consider merge operations within the model of a cooperative game with transferable utility. A *cooperative game* (or game for short) is a tuple (N, v) where N is a finite set consisting of *players*, and v, the so-called *characteristic function*, is a real-valued function on the set of subsets of N, denoted by 2^N, with the convention that the empty set $\emptyset$ is valued 0. Subsets of N are called *coalitions*, and the *value* $v(S)$ of a coalition S denotes the gain the players in S can obtain by joining their resources.

For each $T \subseteq N$, $T \neq \emptyset$, let (N, u_T) denote the so-called *unanimity game*, associated with the coalition T, defined by $u_T(S) = 1$ if all players of T are members of S, and 0 otherwise.

The set of games with player set N, denoted by G^N, may be considered to be equal to the $(2^{|N|} - 1)$-dimensional Euclidean vector space $\mathbb{R}^{2^N \setminus \{\emptyset\}}$. It is well-known that the unanimity games form a basis of G^N [see Shapley (1953)]. The unique weights $\Delta^N(v,T)$, $T \subseteq N$ non-empty, of a game (N,v) such that

$$v = \sum_{T\subseteq N, T\neq\emptyset} \Delta^N(v,T) u_T\,, \tag{1}$$

are called the (Harsanyi-)*dividends*. From (1) it follows that the value $v(S)$ equals the sum $\sum_{T\subseteq S, T\neq\emptyset} \Delta^N(v,T)$ of dividends of all sub-coalitions of the coalition S, and from this one easily derives the well-known recursive formula for the dividends:

- $\Delta^N(v,S) = v(S)$ for one-person coalitions S, and
- $\Delta^N(v,S) = v(S) - \sum_{T\subset S, T\neq\emptyset} \Delta^N(v,T)$ for multi-person coalitions S.

Based on this property the dividend of a coalition is sometimes interpreted as that part of the coalition value that is achieved by the cooperation of all players in the coalition, none excluded.

One of the main goals in cooperative game theory is to examine rules or *solution concepts* for the distribution of the value of the grand coalition N among the players. The most well-known single-valued rule is the *Shapley value* ϕ. In terms of dividends it distributes all dividends equally among the involved players:

$$\phi_i(N,v) = \sum_{T\subseteq N, i\in T} \frac{\Delta^N(v,T)}{|T|} \quad \text{for each } i \in N\,.$$

3. Profitability in Merge Situations

The Shapley value will be discussed here in the context of merging. Let $\Pi = (T_j)_{j\in J}$ be a partition of N (the coalitions in Π are assumed to be non-empty), and let us suppose that each player is only interested in a coalition S if all her "associates" in T_j are involved: $T_j \subseteq S$, with $j \in J$ such that $i \in T_j$. Therefore, only unions of coalitions in Π are considered, so that the game (J, v^Π) is actually played, with

$$v^\Pi(I) = v(\cup_{j\in I} T_j) \quad \text{for each non-empty } I \subseteq J, \text{ and } v^\Pi(\emptyset) = 0\,.$$

This game is also called the *quotient game* of v w.r.t. the partition Π, and it evolves from the situation where the players in each compartment of Π merge into one player.

Let us further assume that players in games are rewarded according to the Shapley value. Merging w.r.t. the partition $\Pi = (T_j)_{j\in J}$ is called *profitable* for a compartment T_j of Π if

$$\phi_j(J, v^\Pi) \geq \sum_{i\in T_j} \phi_i(N,v) \tag{2}$$

holds. And if (2) holds for all $j \in J$, with $|T_j| > 1$, we say that it is *profitable* to merge w.r.t. Π. Observe that in this case the net gain of the merging players has to be equal to the net loss of the non-merging players, due to the fact that the grand coalition values of both games (N,v) and (J,v^Π) are equal.

By Π/T we denote the partition of coalition T, induced by Π, i.e. $\Pi/T = \{T \cap T_j : j \in J \text{ and } T \cap T_j \neq \emptyset\}$; J/T denotes the index set $\{j \in J : T \cap T_j \neq \emptyset\}$. The average number of players of T in the relevant compartments of the partition Π is denoted by $\gamma(\Pi, T)$, i.e. $\gamma(\Pi, T) = |T|/|J/T|$. It is called the *average presence* of T in Π. The following theorem supplies sufficient conditions on the game so that merge profitability will occur.

Theorem 3.1. *It is profitable in a game (N, v) to merge w.r.t. a partition $\Pi = (T_j)_{j \in J}$ for a coalition T_j if for all coalitions T, with $T_j \cap T \neq \emptyset$, we have*

$$\begin{aligned} \gamma(\Pi, T) &\geq |T_j \cap T| \quad \text{if } \Delta^N(T, v) > 0, \quad \text{and} \\ \gamma(\Pi, T) &\leq |T_j \cap T| \quad \text{if } \Delta^N(T, v) < 0 . \end{aligned} \tag{3}$$

In words, it is profitable for T_j to merge if each coalition, containing players of T_j, does not have more players in T_j than the average presence of this coalition in Π if it has a positive dividend, and not fewer players if its dividend is negative.

Proof. Given a fixed player set and a fixed partition, it is evident that the transformation of games on this player set into quotient games w.r.t. the partition is a linear function. Hence, we have

$$(J, v^{\Pi}) = \sum_{T \subseteq N,\ T \neq \emptyset} \Delta^N(v, T)(J, (u_T)^{\Pi}) .$$

The game $(J, (u_T)^{\Pi})$ is actually the unanimity game $(J, u_{J/T})$ since $(u_T)^{\Pi}(I) = u_T(\cup_{j \in I} T_j) = 1$ if and only if $T \subseteq \cup_{j \in I} T_j$, and the latter is clearly equivalent to $J/T \subseteq I$.

It is well-known that the Shapley value is a linear function, so that we have

$$\begin{aligned} \phi_j(J, v^{\Pi}) &= \sum_{T \subseteq N,\ T \neq \emptyset} \Delta^N(v, T) \phi_j(J, u_{J/T}) \\ &= \sum_{T \subseteq N,\ T \cap T_j \neq \emptyset} \frac{\Delta^N(v, T)}{|J/T|} . \end{aligned}$$

On the other hand, the players in T_j in the original game (N, v) receive the amount

$$\begin{aligned} \sum_{i \in T_j} \phi_i(N, v) &= \sum_{i \in T_j} \sum_{T \subseteq N, i \in T} \frac{\Delta^N(v, T)}{|T|} \\ &= \sum_{T \subseteq N,\ T \cap T_j \neq \emptyset} \Delta^N(v, T) \frac{|T \cap T_j|}{|T|} . \end{aligned}$$

Observe that the conditions in the theorem imply

$$\frac{\Delta^N(v, T)}{|J/T|} \geq \Delta^N(v, T) \frac{|T \cap T_j|}{|T|}$$

for all coalitions T with $T \cap T_j \neq \emptyset$, and from this we conclude that $\phi_j(J, v^{\Pi}) \geq \sum_{i \in T_j} \phi_i(N, v)$ must hold, so that it is profitable for T_j to merge. $\square$

The above proof is mainly based on the linearity property of the Shapley value. Other linear values like the weighted Shapley values [see Monderer *et al.* (1992)], the sharing values [see Derks *et al.* (1998)], and the Owen value [see Owen (1977) and Vázquez-Brage *et al.* (1997)] easily fit into this context, and therefore, similar results may be expected for the mentioned distribution concepts as well.

The question is now how Theorem 3.1 may be of help for players in search for coalitions which are interesting from the point of merging. We say that T is *mergeable* if it is profitable to merge w.r.t. the so-called T-partition $\Pi(T)$, consisting of the compartments T and the one-person coalitions of players outside T. A T-partition corresponds to the situation where only the players in T merge. Theorem 3.1 gives us now sufficient conditions for a coalition to be mergeable. Suppose that $\Pi(T)$ fulfills the conditions mentioned in the theorem. For a coalition S, with $S \cap T \neq \emptyset$, we have

$$\gamma(\Pi(T), S) = \frac{|S|}{1 + |S \backslash T|} = \frac{|S \cap T| + |S \backslash T|}{1 + |S \backslash T|} \leq |S \cap T| \frac{1 + |S \backslash T|}{1 + |S \backslash T|} = |S \cap T| .$$

This implies that, in view of the first statement of condition (3), equality should hold for coalitions S with positive dividend, and this is only possible for the cases $S \subseteq T$ or $|S \cap T| = 1$. From this we derive the following result.

Theorem 3.2. *A coalition T is mergeable in a game (N, v) if each coalition with positive dividend is either contained in T or has at most one player in common with T.*

Observe that a player can try to find a mergeable coalition by simply excluding all players with whom he is participating in a coalition with a positive dividend. In particular, if there is another player so that all coalitions have a non-positive dividend if both players are members, then both players form a mergeable coalition.

Example 3.1. Let N consist of at least five players, and consider the game (N, u), with $u(S) = |S| - 1$ for non-empty $S \subseteq N$. Consider the situation that coalition T merges, $T \neq N$ and $|T| > 1$. The game (N', u') after merging looks like $N' = N \backslash T \cup \{t\}$ where t represents the merged coalition T, and $u'(S) = u(S)$ if $t \notin S$, and $u'(S) = u(S \backslash \{t\} \cup T) = (|S| - 1 + |T|) - 1$ if $t \in S$. Thus, $(N', u') = (N', u) + (N', (|T| - 1) u_{\{t\}})$. The sum of the Shapley values of the players in T in the symmetric game (N, u) equals $|T| u(N)/|N| = |T| - |T|/|N|$ whereas the Shapley value of the player t in the game (N', u') equals $u(N')/|N'| + |T| - 1 = |T| - 1/(|N| - |T| + 1)$, which is larger. We conclude that all coalitions are mergeable in (N, u).

The previous example also shows that Theorem 3.1 cannot be reversed: the dividends of this game are $\Delta^N(S, v) = (-1)^{|S|}$ if $|S| > 1$, and $\Delta^N(S, v) = 0$ for the one-person coalitions S. Let Π be a non-trivial partition so that there exist two multi-person coalitions, say T' and T'' in Π. Take a coalition T with an odd number of players and $|T' \cap T| = 1$ and $|T'' \cap T| > 1$. Then $\Delta^N(T, v) = -1 < 0$, and $\gamma(\Pi, T) = |T|/|J/T| > 1 = |T' \cap T|$; therefore, the conditions in Theorem 3.1 are not fulfilled, and we conclude that there are no partitions with two or more multi-person compartments, such that these conditions hold.

On the other hand, there exist partitions with multi-person compartments such that it is profitable to merge w.r.t. these partitions; for example, let there be an even number of players and suppose the partition $\Pi = (T, N\backslash T)$ split the player set in two equal-sized multi-person compartments. Then, because of symmetry, T and $N\backslash T$ both obtain half of the value of grand coalition in the quotient game, and this obviously equals $v(N)$. Also, the Shapley value assigns half of $v(N)$ to the players in T and in $N\backslash T$, so that (2) is fulfilled for each compartment (with equality).

4. Applications

The applicability of Theorem 3.1 is apparent in situations where the players are aware of the coalition values, and especially of the dividend values that are derived from these values (see Sec. 2). Also in situations, where the players are only aware of some properties of the game they play, the theorem may be of help. Two corollaries and three propositions are provided that can be used in situations where a structure on the set of coalitions with non-zero dividend is known. Further, we will discuss examples of real life instances that lead to games on which our results are applicable.

The proof of the following corollary is easily derived from Theorem 3.1.

Corollary 4.1. *Let (N, v) be a game and $\Pi = (T_j)_{j \in J}$ a partition of N such that $|T \cap T_j| \leq 1$ for all $j \in J$ and coalitions T with $\Delta^N(T, v) \neq 0$. Then $\phi_j(J, v^\Pi) = \sum_{i \in T_j} \phi_i(N, v)$ for each $j \in J$.*

Although there is no real utility profit from merging in the situation of the corollary, we can still speak of a profitable merge operation since the computational complexity of the game is drastically lowered, and furthermore, the strategic properties of the resulting quotient game is much easier to survey.

Let us call a collection Π of coalitions *equal-sized* if all coalitions in Π have equal cardinality. Then Corollary 4.1 can be considered as a special case of

Corollary 4.2. *Let (N, v) be a game and $\Pi = (T_j)_{j \in J}$ a partition of N such that Π/T is equal-sized for all coalitions T with $\Delta^N(T, v) \neq 0$. Then $\phi_j(J, v^\Pi) = \sum_{i \in T_j} \phi_i(N, v)$ for each $j \in J$.*

Proof. For coalition T with $\Delta^N(T, v) \neq 0$, it holds that the average presence $\gamma(\Pi, T)$ equals the number of players in each of the compartments of Π/T, and therefore $\gamma(\Pi, T) = |T_j \cap T|$ for each $j \in J/T$. Applying Theorem 3.1 the corollary now follows. □

Now consider a game where only coalitions with at most two players may have a non-zero dividend. A game with this property is also called a 2-game. It follows easily from Theorem 3.2 that

Proposition 4.1. *All coalitions in a 2-game are mergeable.*

In Van den Nouweland *et al.* (1996), it is proved that for 2-games with only non-negative dividends, the Shapley value, nucleolus and τ-value coincide. So, for these latter two values, merging is also profitable.

In the sequel we will examine two "phone calls"-examples. Merge features in this context were actually noted during the preparation of Van den Nouweland *et al.* (1996). The article reports on a project on the distribution of costs and profits among a group of phone companies in Western Europe. The data of the Dutch phone company, however, were only allowed to be used within the project, so that in order to meet this restriction and the need for a real life instance, the authors decided to use accumulated data. It appeared that the companies, who's data were accumulated, did not lose or gain from this merging.

Example 4.1. (direct phone calls) Consider the situation where each player $i \in N$ is a phone company with their own set of clients. We assume that the net profit a_{ij} is known, for each pair $i, j \in N$, i.e. the profit generated by calls of clients of i to clients of j minus the costs for setting up and maintaining the network system for these calls. The problem of dividing the net profit among the phone companies can be translated to a cooperative game setting by considering the game (N, v) where the value of a non-empty coalition S is defined by $v(S) = \sum_{i,j \in S} a_{ij}$. The dividends of (N, v) equal a_{ii} for the one-person coalitions $\{i\}$, $a_{ij} + a_{ji}$ for the two-person coalitions $\{i, j\}$, and 0 otherwise. We conclude that the above situation evolves into a 2-game, implying that players can profitably merge with each other according to Proposition 4.1.

Now consider the situation where the player set N consists of players of different types, and let T_j denote the set of players of type j, $j \in J$. Suppose that gains are generated only in groups of players in which each type is represented by at most one player. A game (N, v) with this structure is called a *component* game, and has the following form $v = \sum_{T:|T \cap T_j| \leq 1, j \in J} c_T u_T$. So, the non-zero dividends of a component game correspond to coalitions with at most one player of each type. The following proposition is easily derived from Theorem 3.2.

Proposition 4.2. *Each coalition of players of the same type in a component game is mergeable.*

Example 4.2. (indirect phone calls) Gibbens *et al.* (1991) consider the allocation problem of the net profit of the international phone calls from a given home country to a given destination country. A phone call may be handled by phone companies in both countries directly or, if capacity constraints are met, via a phone company in another part of the world. So, each call needs several phone companies: a carrier $i \in N_1$ in the home country, possibly a transit carrier $j \in N_2$, and a carrier $k \in N_3$ in the destination country. Consider the players to be the phone companies around the world. So three types of players exist in this instance, and we assume that profit, that is generated by any combination of carriers, is fixed and independent of other combinations. If the net profit of calls from i to k (via j) is denoted by a_{ik} (a_{ijk}), then this leads to the game (N, v), with $N = N_1 \cup N_2 \cup N_3$ and

$$v(S) = \sum_{i \in S \cap N_1, k \in S \cap N_3} a_{ik} + \sum_{i \in S \cap N_1, j \in S \cap N_2, k \in S \cap N_3} a_{ijk} \,.$$

One easily derives that the dividends of this game equal zero for coalitions T with $|T \cap N_m| > 1$ for $m = 1, 2$ or 3. Therefore, the above situation leads to a component game, and applying Proposition 4.2 it is profitable for phone companies of the same type to merge.

Related to the component games are the decomposable games. Decomposable games occur frequently in real life situations, and especially where some independence exists among the profit gain of two or more distinct groups of players. A game (N, v) is called *decomposable* if there exists a partition $\Pi = (T_j)_{j \in J}$ of N such that $v(S) = \sum_{j \in J} v(S \cap T_j)$ for each coalition S. Let us call Π a *decomposition* of (N, v) and the compartments of Π *subcarriers*. It is easily checked that coalitions with non-zero dividends are subsets of one of the subcarriers, so that mergeable coalitions can easily be traced by the help of Theorem 3.2.

Proposition 4.3. *In a decomposable game, coalitions are mergeable if they have at most one player in common with all subcarriers of a decomposition.*

We conclude with the observation that examples in the literature exists of characterisations of classes of games, like the airport cost games and assignment games, in terms of properties on the collections of coalitions with non-zero dividend. The results in this paper may be of some use in those cases. In this context, we would like to mention a recent result in Derks *et al.* (2000) of a characterisation of games where the Weber set and the Selectope coincide, in terms of a condition on the collections of the positive dividend coalitions and of the negative dividend coalitions. For example, A sufficient condition, for the coincidence of the Weber set and the Selectope, is the existence of only one player who is member of coalitions with non-zero dividends of both signs. According to Theorem 3.2, this player can form mergeable coalitions with any combination of the other players who are only members of negative dividend coalitions.

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